 DRAFT: January 28, 2005

CHAPTER 4: INSTRUCTIONAL STRATEGIES

No single method of instruction is the best or most appropriate in all situations. Teachers have a wide choice of instructional strategies for any given lesson. Teachers might use, for example, direct instruction, investigation, classroom discussion and drill, small groups, individualized formats, and hands-on materials. Good teachers look for a fit between the material to be taught and strategies to teach it. They ask, What am I trying to teach? What purposes are served by different strategies and techniques? Who are my students? What do they already know? Which instructional techniques will work to move them to the next level of understanding? Drawing on their experience and judgment, teachers should determine the balance of instructional strategies most likely to promote high student achievement, given the mathematics to be taught and their students’ needs.

The *Mathematics Content Standards* and this framework include a strong emphasis on computational and procedural competencies as a component of the overall goals for mathematics proficiency. The teaching of computational and procedural skills has its hazards, however. First, it is possible to teach computational and procedural skills in the absence of understanding. This possibility must be precluded in an effective mathematics program. A conceptual understanding of when the procedure should be used, what the function of that procedure is, and how the procedure manipulates mathematical information, provides necessary constraints on the appropriate use of procedures and for detecting when procedural errors have been committed (Geary, Bow-Thomas, and Yao 1992; Ohlsson and Rees 1991).

Students gain a greater appreciation of the essence of mathematics if they are taught to apply mathematical skills to the solution of problems. They may start to

 DRAFT: January 28, 2005

solve problems when armed with a small number of addition facts, and they need not wait to master all addition facts as a prerequisite to problem solving.

In a standards-based curriculum, good lessons are carefully developed and are designed to engage all members of the class in learning activities focused on student mastery of specific standards. Such lessons connect the standards to the basic question of why mathematical ideas are true and important. Central to the *Mathematics Content Standards* and this framework is the goal that all students will master all strands of the standards. Lessons will need to be designed so that students are constantly being exposed to new information while practicing skills and reinforcing their understanding of information introduced previously. The teaching of mathematics does not need to proceed in a strict linear order, requiring students to master each standard completely before being exposed to the next, but it should be carefully sequenced and organized to ensure that all standards are taught at some point and that prerequisite skills form the foundation for more advanced learning. Practice leading toward mastery can be embedded in new and challenging problems.

A particular challenge that the standards present to educators and publishers is the instruction of grade-level topics for students who have not yet mastered the expected content for earlier grades. One approach is to focus on the more important standards, as noted in Chapter 3, “Grade-Level Considerations.” Bringing students up to grade-level expectations for those areas of emphasis will likely require (1) additional classroom time for mathematics, including time before, during, or after the instructional day; (2) the identification of the component skills that comprise each of the areas of emphasis; and (3) a reliable and valid means of assessing the degree to which individual students have mastered the component skills.

Instructional resources that help teachers to identify easily where these component skills were introduced in previous grades and that allow teachers to

 DRAFT: January 28, 2005

adapt these earlier-grade-level units into “refresher” lessons will be helpful. The development of such resources might require the development of a master guide on the organization of instructional units across grade levels. To achieve the goal of bringing students up to grade-level expectations requires that instructional materials be well integrated across grade levels. For example, instructional materials for fourth grade students should be written not only to address the fourth grade standards but also to prepare the foundation for mastery of later standards.

Organization of This Chapter

To guide educators in designing instructional strategies, this chapter is organized into three main sections:

1. “Instructional Models: Classroom Studies” provides an overview of research on student learning in classroom settings. In this section Table 1, “Three-Phase Instructional Model,” provides a simple, research-based approach to instruction that all teachers may use.
2. “Instructional Models: View from Cognitive Psychology” provides a description of the research in cognitive psychology on the mechanisms involved in learning.
3. “General Suggestions for Teaching Mathematics” describes ways in which to organize the teaching of mathematics in kindergarten through grade twelve. Table 2, “Outline for Instruction of School-Based Mathematics,” provides a convenient summary of the most important considerations for developing good lesson plans.

Instructional Models: Classroom Studies

Although the classroom teacher is ultimately responsible for delivering instruction, research on how students learn in classroom settings can provide useful information

 DRAFT: January 28, 2005

to both teachers and developers of instructional resources. This section provides an overview of student learning in classroom settings.

In conjunction with the development of this framework and the *Mathematics Content Standards,* the California State Board of Education contracted with the National Center to Improve the Tools of Educators, University of Oregon, in Eugene, to conduct a thorough review of high-quality experimental research in mathematics (Dixon et al. 1998). The principal goal of the study was to locate high-quality research about achievement in mathematics, review that research, and synthesize the findings to provide the basis for informed decisions about mathematics frameworks, content standards, and mathematics textbook adoptions.

From a total of 8,727 published studies of mathematics education in elementary and secondary schools, the research team identified 956 experimental studies. Of those, 110 were deemed high-quality research because they met tests of minimal construct and internal and external validity. The test of minimal construct looked at whether or not the study used quantitative measurements of mathematics achievement to report the effects of an instructional approach. To meet the internal validity criterion, the study had to use a true experimental design, have sufficient information to compute effect sizes, have equivalencies of groups at pretest, and use a representative and unbiased sample. External validity looked at whether or not the approach was implemented in settings representative of actual instructional conditions. The original report that the research team presented to the State Board of Education contained reviews of 77 of the qualifying studies; the most recent report includes information from all 110.

The reviewers cautioned readers about what their review did not do. Although a goal of the study was to find experimental support for the scope of instruction and the sequence of instructional topics, none of the high-quality experimental research studies addressed these important aspects of mathematics instruction. Instead, they

 DRAFT: January 28, 2005

looked only at findings relating to mathematics achievement. In addition, the review did not address such areas as improved attitudes toward mathematics or preferences for one mode of instruction over another.

Studies that met the high-quality review criteria indicated clear and positive gains in achievement from some types of instructional strategies. Perhaps most important, the review indicated marked differences in the effects of “conventional mathematics instruction” contrasted with interventions associated with high student achievement.

Two-Phase Model

As defined in the review, conventional mathematics instruction followed a two-phase model. In the first phase the teacher demonstrated a new concept, algorithm, or mathematical strategy while the students observed. In the second phase the students were expected to work independently to apply the new information, often completing work sheets, while the teacher might (or might not) monitor the students’ work and provide feedback. This two-phase model, the researchers noted, was characterized by an abrupt shift in which students were expected “to know and independently apply the information newly taught moments earlier” (Dixon et al. 1998).

Three-Phase Model

More effective strategies may incorporate a variety of specific techniques, but they generally follow a clear three-phase pattern, as shown in Table 1, “Three-Phase Instructional Model.”

The first phase. In the first phase the teacher introduces, demonstrates, or explains the new concept or strategy, asks questions, and checks for understanding. The students are actively involved in this phase instead of simply observing the teacher’s lecture or demonstration. *Actively involved* should be thought of as a

 DRAFT: January 28, 2005

necessary but not sufficient characteristic of the first phase of effective mathematics instruction. (It is easy to imagine students actively involved in initial instructional activities that do not directly address the skills, concepts, knowledge, strategies, problem-solving competence, and understanding specified in the *Mathematics Content Standards.*) One way or another, all students must be involved actively in the introduction of new material. The corollary is that no student should be allowed to sit passively during the introduction of new material. The understanding demonstrated by a few students during this phase of instruction does not guarantee that all students understand. Teachers' classroom management and instructional techniques and the clarity and comprehensibility of initial instruction contribute to the involvement of all students. At the very least, active participation requires that the student attend to, think about, and respond to the information being presented or the topic being discussed.

The second phase. The second phase is an intermediate step designed to result in the independent application of the new concept or described strategy. This second step—the "help phase"—occurs when the students gradually make the transition from "teacher-regulation" to "self-regulation" (Belmont 1989). The details and specific instructional techniques of this phase vary considerably, depending on the level of student expertise and the type of material being taught. These techniques include any legitimate forms of prompting, cueing, or coaching that help students without making them dependent on pseudo-help crutches that do indeed help students but are not easily discarded. During this phase teachers also informally, but steadfastly, monitor student performance and move more slowly or more quickly toward students' independent, self-regulated achievement according to what the monitoring reveals about the students' progress.

The third phase. In the third phase students work independently. In contrast with conventional lessons, however, the third phase is relatively brief instead of taking up

 DRAFT: January 28, 2005

most of the lesson time. This phase often serves in part as an assessment of the extent to which students understand what they are learning and how they will use their knowledge or skills in the larger scheme of mathematics.

This three-phase model is not rigid. If students do not perform well during the guided phase of instruction, then teachers should go back and provide additional clear and comprehensible instruction. If students do not perform well when they work independently, they should receive more guided practice and opportunities for application. And finally, if students perform well on a given topic independently but later display weaknesses with respect to that topic, then teachers should return to further guided instruction. This method is particularly critical when the topic at hand is clearly a prerequisite to further mathematics instruction and skill.

The table that follows shows each phase of the three-phase instructional model.

(Insert Table here)

This three-phase model is framed by a beginning point (central focus) and by an ending point (closure), both of which are discussed next.

Central focus. In planning their lessons, teachers need to begin by identifying a central focus—the lesson’s specific mathematical content and the goal of the lesson or sequence of lessons. Teachers also need to address the following concerns:

- The lesson or series of lessons should be focused on a clear instructional goal that is related to the mathematical content of the standards.
- The goal will typically be focused on fostering students’ computational and procedural skills, conceptual understanding, mathematical reasoning, or some combination of these.

 DRAFT: January 28, 2005

- The focus of a lesson or series of lessons is not simply to “cover” the required material but to build on previous knowledge and to prepare for future learning. Ultimately, the goals of any lesson are understood in the context of their relation to grade-level content, content covered in earlier grades, and content to be covered in later grades.

Closure. Closure of a lesson may take many forms. At the end of each lesson or series of lessons, students should not be left unsure of what has been settled and what remains to be determined. Whether the topic is covered within a single lesson or many, each lesson should contain closure that ties the mathematical results of the activities to the central goal of the lesson and to the goals of the overall series of lessons.

While current and confirmed research such as that reported in the Dixon study provides a solid basis on which to begin to design instruction, research from cognitive psychology provides insights into when and how children develop mathematical thinking.

Instructional Models: View from Cognitive Psychology

Initial competencies for natural abilities are built into the mind and brain of the child. These competencies develop during the child’s natural social and play activities. Academic learning involves training the brain and mind to do what they were not designed by nature to do without help.

Natural Learning

The development of oral language is one example of natural learning. Young children naturally learn to speak as they listen to the speech around them. By the time they are five years old, they understand and can use approximately 6,000 to 15,000 words; they speak in coherent sentences using the basic conventions of the

 DRAFT: January 28, 2005

spoken language around them; and they can communicate effectively. Another example of natural learning is the early development of understanding about numbers. Starkey (1992) tested young children to determine their early understanding of arithmetic. He put up to three balls into a “search box” (a nontransparent box into which things are dropped and retrieved). At the age of twenty-four months (and sometimes younger), children would drop three balls into the box and then retrieve exactly three balls and stop looking for other balls in the box, showing that they were able to represent the number three mentally. They did this task without verbalizing, suggesting that a basic understanding of arithmetic is probably independent of language skills (Geary 1994, 41).

Certain features of geometry appear to have a natural foundation (Geary 1995). People know how to get from one place to another; that is, how to navigate in their environment. Being able to navigate and develop spatial representations, or cognitive maps, of familiar environments is a natural ability (e.g., picturing the location of the rooms in a house and the furniture in them). Without effort or even conscious thought, people automatically develop rough cognitive maps of the location of things in familiar environments, both small-scale environments, such as their house, and large-scale environments, such as a mental representation of the wider landscape (in three dimensions). Children’s play, such as hide-and-seek, often involves spatial-related activities that allow children to learn about their environment without knowing they are doing so (Matthews 1992).

The brain and cognitive systems that allow us to navigate include an implicit understanding of basic Euclidean geometry. For example, we all implicitly know that the fastest way to get from one place to another is to “go as the crow flies”; that is, in a straight line. This is an example of natural conceptual knowledge. It is also sometimes taken as the first postulate of Euclidean geometry in school textbooks: A straight line can be drawn between any two points.

 DRAFT: January 28, 2005

Academic Learning

The human brain and mind are biologically prepared for an understanding of language and basic numerical concepts. Without effort, children automatically learn the language they are exposed to, they develop a general sense of space and proportion, and they understand basic addition and subtraction with small numbers. Their natural social and play activities ensure that they get the types of experiences they need to acquire these fundamental skills. Not all cognitive abilities develop in this manner, however. In fact, most academic, or school-taught, skills do not develop in this manner because they are in a sense “unnatural” or formally learned skills (Geary 1995). As societies become more technically complex, success as an adult, especially in the workplace and also at home (e.g., managing one’s money), involves more academic learning—skills that the brain and mind are not prewired to learn without effort. It is in those societies that academic schooling first emerged.

Schools organize the activities of children in such a way that they learn skills and knowledge that would not emerge as part of their natural social and play activities (Geary et al. 1998). If this were not the case, schooling would be unnecessary. But schooling is necessary, and it is important to understand why. Schooling is not necessary for the development of natural learning but is absolutely essential for academic learning. This is why teaching becomes so important. Teachers and instructional materials provide the organization and structure for students to develop academic skills, which include most academic domains; whereas nature provides for natural abilities. For academic domains this organization often requires explicit instruction and an explicit understanding of what the associated goals are and how to achieve them.

There are important differences in the source of the motivation for engaging in the activities that will foster the development of natural and academic abilities (Geary

 DRAFT: January 28, 2005

1995). Children are biologically motivated to engage in activities, such as social discourse and play, that will automatically—without effort or conscious awareness—flesh out natural abilities, such as language. The motivation to engage in the activities that foster academic learning, in contrast, comes from the increasingly complex requirements of the larger society, not from the inherent interests of children. Natural play activities, or natural curiosity, of school-age students cannot be seen as sufficient means for acquiring academic abilities, such as reading, writing, and much of mathematics. The interests, likes, and dislikes of children are not a reliable guide to what is taught and how it is taught in school, although the interests of children probably can be used in some instructional activities. Once basic academic abilities are developed, natural interests can be used to motivate further engagement in some, but probably not all, academic activities.

Also relevant is intellectual curiosity, an important dimension of human personality (Goldberg 1992). People with a high degree of intellectual curiosity will seek out novel information and will often pursue academic learning on their own. Nevertheless, there are large individual differences in curiosity and, in fact, all other dimensions of personality. Some students will be highly curious and will actively seek to understand many things; others will show very little curiosity about much of anything; and most will be curious about some things and not others. If the goal is that all students meet or exceed specific content standards, then teachers cannot rely on natural curiosity to motivate all children to engage in academic learning.

In summary, natural mathematical abilities include the ability to determine automatically and quickly the number of items in sets of three to four items and a basic understanding of counting and very simple addition and subtraction; for example, that adding increases quantity (Geary 1995). These skills are evident in human infants and in many other species. Certain features of geometry, and

DRAFT: January 28, 2005

perhaps statistics, also appear to have a natural foundation, although indirectly (Brase, Cosmides, and Tooby 1998).

Much of the content described in the *Mathematics Content Standards* is, however, academic. Mastering this content is essential for full participation in our technologically complex society; but students are not biologically prepared to learn much of this material on their own, nor will all of them be inherently motivated to learn it. That is why explicit and rigorous standards, effective teaching, and well-developed instructional materials are so important. The *Mathematics Content Standards,* teachers, and well-designed textbooks must provide for students' mathematical learning; that is, an understanding of the goals of mathematics, its uses, and the associated procedural and conceptual competencies.

General Suggestions for Teaching Mathematics

A general outline for approaching the instruction of school-based mathematics is presented in Table 2, "Outline for Instruction of School-Based Mathematics." Here, the teaching of mathematical units is focused on fostering the student's understanding of the goals of the unit and the usefulness of the associated competencies and on fostering general procedural and conceptual competence.

It is important to tell students the short-term goals and sketch the long-term implications of the mathematics they are expected to learn and the contexts within which the associated competencies, when developed, can be used. The short-term goals usually reflect the goal for solving a particular class of problem. For example, one goal of simple addition is to "find out the sum of two groups when they are put together." Knowing the goal of problem solving appears to facilitate the development of procedural and conceptual problem-solving competencies (Siegler and Crowley 1994).

 DRAFT: January 28, 2005

Students should also be told some of the longer-term goals of what they are learning. This might include (1) stating what the students will be able to do at the end of the unit, semester, or academic year in relation to the mathematics standards; and (2) clarifying how current learning relates to the mathematics students will learn in subsequent years. It is also helpful to point out some of the practical uses of the new skills and knowledge being learned by linking them to careers and personal situations. *Studies of high school students indicate that making the utility of mathematics clear increases the student's investment in mathematical learning; that is, increases the number of mathematics courses taken in high school* (Fennema et al. 1981).

The usefulness of newly developing competencies might also be illustrated by having students use their skills in real-world simulations or projects; for example, figuring out how much four items cost at a store or using measurement and geometry to design a tree house. Assigning projects would not be the usual route to developing these competencies but would be a means of demonstrating their usefulness and providing practice. Projects might be used to introduce a difficult concept or to engage students in the unit. Using projects to stimulate interest and involvement must be weighed against the time they require and the extent of the mathematics learning. *Long projects with limited mathematical content and learning should be avoided.*

Procedural and Conceptual Competencies in Mathematics

Chapter 1, "Guiding Principles and Key Components of an Effective Mathematics Program," notes that the development of mathematical proficiency requires both procedural skills and conceptual knowledge and that these two components of mathematical competency are interrelated. It is now understood that the same activities—such as solving problems—can foster the acquisition of procedural skills and conceptual knowledge and can lead to the use of increasingly sophisticated problem-solving strategies (Siegler and Stern 1998; Sophian 1997). At the same time research in cognitive psychology suggests that different types of instructional activities will favor the development of procedural competencies more than conceptual knowledge, and other types of instructional activities will favor the development of conceptual knowledge more than procedural competencies (Cooper and Sweller 1987; Sweller, Mawer, and Ward 1983; Geary 1994).

(See figure 1, "The Components of Conceptual, Procedural, and Reasoning Skills.")

Fostering procedural competencies. The learning of mathematical procedures, or algorithms, is a long, often tedious process (Cooper and Sweller 1987). To remember mathematical procedures, students must practice using them. Students should also practice using the procedure on all the different types of problems for which the procedure is typically used. Practice, however, is not simply solving the same problem or type of problem over and over again. Practice should be provided in small doses (about 20 minutes per day) and should include a variety of problems (Cooper 1989).

These arguments are based on studies of human memory and learning that indicate that most of the learning occurs during the early phases of a particular practice session (e.g., Delaney et al. 1998). In other words, for any single practice session, 60 minutes of practice is not three times as beneficial as 20 minutes. In fact, 60 minutes of practice over three nights is much more beneficial than 60 minutes of practice in a single night.

Moreover, it is important that the students not simply solve one type of problem over and over again as part of a single practice session (e.g., simple subtraction problems, such as 6 – 3, 7 – 2). This type of practice seems to produce only a rote use of the associated procedure. One result is that when students attempt to solve a somewhat different type of problem, they tend to use, in a rote manner, the procedure they have practiced the most, whether or not it is applicable. For example,

 DRAFT: January 28, 2005

one of the most common mistakes young students make in subtraction is to subtract the smaller number from the larger number regardless of the position of the numbers. The problem shown below illustrates this type of error, which will be familiar to most elementary school teachers:

$$\begin{array}{r} 42 \\ \underline{-7} \\ 45 \end{array}$$

Practicing the solving of simple subtraction problems (e.g., 7 – 2) is important in and of itself. Unthinking or rote application of the procedure is not the only cause of this type of error. In fact, this type of error should be a red flag for the teacher because it probably reflects the student’s failure to understand regrouping.

One way to reduce the frequency of such procedural errors is to have the students practice problems that include items requiring different types of procedures (e.g., mixing subtraction and addition problems and, if appropriate, simple and complex problems). This type of practice provides students with an opportunity to understand better how different procedures work by making them think about which is the most appropriate procedure for solving each problem.

Ultimately, students should be able to use the procedure automatically on problems for which the procedure is appropriate. *Automatically* means that the procedure is used quickly and without errors and without the students having to think about what to do. Extensive practice, distributed over many sessions across many months or even years, might be needed for students to achieve automaticity for some types of mathematical algorithms.

Research indicates that long-term (over the life span) retention of mathematical competencies (and competencies in other areas) requires frequent refreshers (i.e., overviews and practice) at different points in the students’ mathematical instruction

 DRAFT: January 28, 2005

(Bahrick and Hall 1991). One way to provide such a refresher is through a brief overview of related competencies when the students are moving on to a more complex topic.

In general, refreshers should focus on those basic or component skills needed to successfully solve new types of problems. For example, skill at identifying equal fractions, along with a conceptual understanding of fractions, will make learning to reduce fractions to their lowest terms much easier. These refreshers will provide the distributed practice necessary to ensure the automatic use of procedures for many years after the students have left school.

Fostering conceptual competencies. Fostering students' conceptual understanding of a problem or class of problems is just as important as developing students' computational and procedural competencies. Without conceptual understanding, students often use procedures incorrectly. More specifically, they tend to use procedures that work for some problems on problems for which the procedures are inappropriate.

Conceptual competency has been achieved when students understand the basic rules or principles that underlie the items in the mathematics unit. Students with this level of competency no longer solve problems according to the superficial features of the problem but by understanding the underlying principles. Students with a good conceptual understanding of the material are more flexible in their problem-solving approaches, see similarities across problems that involve the same rule or principle, make fewer procedural errors, and can use these principles to solve novel problems.

A number of teaching techniques can be used to foster students' conceptual understanding of problems (Cooper and Sweller 1987; Sweller, Mawer, and Ward 1983):

 DRAFT: January 28, 2005

- First, when possible, the teacher should try to illustrate the problem by using contexts that are familiar and meaningful to students. In addition to fostering the students' conceptual understanding, familiar contexts will help students to remember what has been presented in class. Word problems, for example, should be presented in such contexts as home, school, sports, or careers.
- Second, after the students have developed some skill in solving this type of problem, the teacher should present a few problems that are good examples of the type of problem being covered and have the students solve the problems in a variety of ways. Psychological studies have shown that solving a few problems in many different ways is much more effective in fostering conceptual understanding of the problem type than is solving multiple problems in the same way. Solving problems in different ways can be done either as individual assignments or by the class as a whole. In the latter case, different students or groups of students might present suggestions for solving the problem. The students should be encouraged to explain why different methods work and to identify some of the similarities and differences among them.
- Third, for some problems, the teacher might have to teach one approach explicitly and then challenge the class to think of another way to solve the problem. Another approach is to have a student explain to the teacher why the teacher used a certain method to solve a problem (Siegler 1995). Having students explain what someone else was thinking when he or she solved a problem facilitates their conceptual understanding of the problem and promotes the use of more sophisticated procedures during problem solving.

Errors should not simply be considered mistakes to be corrected but an opportunity to understand how the student understands the problem. Extensive studies of mathematical problem-solving errors indicate that most are not trivial but

 DRAFT: January 28, 2005

are systematic (VanLehn 1990). Generally, errors result from confusing the problem at hand with related problems, as in stating $3 + 4 = 12$ (confusing multiplication and addition), or from a poor conceptual understanding of the problem.

Typically, errors will result from confusion of related topics, such as addition and multiplication, a common memory retrieval error even among adults (Geary 1994). For other problems the error will reflect a conceptual misunderstanding, such as confusing the rules for solving one type of problem with those for solving a related type. For example, when students are first learning to solve simple subtraction problems, they are asked to subtract the smaller number from the bigger number (e.g., 6 – 3) so that all of the differences are positive. From these problems, many students form the habit of taking the smaller number away from the larger number when subtracting. This rule works with simple problems but is often inappropriately applied to more complex problems; for example, those with a negative difference, such as 3 – 6, or those that require borrowing, as in 42 – 7. (Appendix B provides a sample East Asian mathematics lesson that can be used in staff development activities to stimulate a more extensive discussion about different ways of teaching mathematics.)

Having students provide several different ways to solve a problem and then spending time focusing on conceptual errors that might occur during this process can take up a significant portion of a lesson, but this method of instruction is usually worth the time. Students can practice using the associated procedures as part of their homework, or practice can occur in school, with a focus on practice and problem solving occurring on alternate days. A good policy is to be sure that the students have a conceptual understanding of the problem before they are given extensive practice on the associated procedures.

One way to monitor conceptual development is to ask students periodically to explain their reasoning about a particular mathematical concept, procedure, or

 DRAFT: January 28, 2005

solution. These explanations may be given either verbally or in writing. They provide a window for viewing the development of the students' understanding. During the early stages, for example, a student may be able to demonstrate a rudimentary grasp of a mathematical concept. At an intermediate stage the student may be able to give an appropriate mathematical formula. Students at advanced stages may be able to present a formal proof. Young students will not always have at their command the correct mathematical vocabulary or symbols, but as they progress, they should be encouraged to use the appropriate mathematical language.

For example, in the following problem students are asked to find the sum of the first n consecutive odd numbers. First one sees a pattern and conjectures at the general formula. The pattern is:

$1 = 1 = 1^2$

$1 + 3 = 4 = 2^2$

$1 + 3 + 5 = 9 = 3^2$

$1 + 3 + 5 + 7 = 16 = 4^2$

and so forth

Young students might explain this pattern in a number of ways. An older student might describe it as a formula. Students at the level of Algebra II might prove it by mathematical induction. (This is just one example of a problem that can be used to assess a student's development of mathematical reasoning.)

Overview and General Teaching Scheme

Figure 2, "General Framework for Teaching a Mathematics Topic," presents a flowchart that might be useful in preparing mathematics lessons for standards-based instruction. It should not be interpreted as a strict prescription of how standards-based instruction should be approached but as an illustration of many possible

 DRAFT: January 28, 2005

sequences of events, although most lessons or series of lessons should attempt to address most of these issues.

First, it is important to introduce the goals and specific mathematics content to be covered, along with some discussion of the different ways in which this type of mathematics is useful. For some topics the mathematics will be directly used in many jobs and even at home (e.g., shopping for the best price), or it might be a building block for later topics.

The next step is to present a brief overview (perhaps one part of a lesson) of the component skills needed to solve the problems to be introduced (e.g., review counting for lessons on addition, review simple addition in preparation for lessons on more complicated addition, and so forth). As part of this review, homework assignments that provide practice in these basic component skills would be useful. In this way the students will receive the extended and distributed practice necessary for them to retain mathematical procedures over the long term. At the same time this refresher will provide continuity from one unit to the next.

It is probably better to teach the conceptual features of the topic before giving the students extensive practice on the associated procedures. This instruction might be provided in three steps, although the number of steps used and the order in which they are presented will vary from one topic to the next. The first step is to introduce the basic concepts (e.g., trading or base-10 knowledge) needed to understand the topic. The second step is to design several lessons that involve solving a few problems in multiple ways and analyzing errors—this step will occur after the students understand the basic concepts and have some competence in solving the class of problem. The third step is to present an overview of the conceptual features of the topic, focusing on those areas where errors were most frequent. Once the initial class discussions of the topic have begun, the students can begin homework assignments (or alternating class assignments) in which they practice solving

DRAFT: January 28, 2005

problems. Portions of these homework assignments can also be used as part of the refresher material for later lessons.

Homework

Student achievement will not improve much without study beyond the classroom. Homework should begin in the primary grades and increase in complexity and duration as students progress through school. To be an effective tool, homework must be a productive extension of class work. Its purpose and connection to class work must be clear to the teacher, student, and parents. The effective use of homework comes into play in Phase 2 of the three-phase instructional model outlined in Table 1. If the teacher chooses to allocate class time to discussion and feedback to students, he or she should ensure that this is productive instructional time for the class, a time when students are analyzing their errors and building their mathematical understanding. Instructional time is precious and should be used wisely. *Using substantial portions of the class period for homework is not an effective use of instructional time.* Using instructional time to review and correct common misconceptions evident from the teacher's analysis of the completed homework or using the last few minutes of a period to make sure that students understand the homework assignments, and how to complete them, can be effective uses of instructional time.

Several types of productive homework are outlined in Table 3.

Homework should increase in complexity and duration as students mature. Students studying for the *Advanced Placement* or *International Baccalaureate* examinations in mathematics will need additional study.